Reversible and irreversible emittance growth

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Emittance growth in bunched charged particle beams in linear accelerators is considered in the case of a Hamiltonian system where nonlinear and time-dependent forces are dominant. Emittance growth is divided into two classes: reversible and irreversible depending on the corresponding entropy change. We consider the case where the measurement resolution is important. We show that a generalized free-energy function acts as a driver for phase-space evolution and emittance growth. [S1063-651X(98)09501-4]

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I. INTRODUCTION

In this paper we investigate the issue of the reversibility of rms emittance growth in Hamiltonian systems. We discuss the emittance growth of bunched beams in the presence of time-dependent nonlinear space-charge and external forces. In particular, we consider the case of nonthermalized relativistic electron beams [1] in linear accelerators where interparticle collisional processes are not important in determining emittance growth. To determine whether emittance growth is reversible or not, we consider the changes in the entropy of the distribution in the case where the resolution of the phase-space measurement (in simulation and in experiment) is of significance. In addition, we consider a generalized free-energy principle, which includes external and internal nonlinear and time varying forces. We derive an expression for a generalized thermodynamic potential, analogous to the Helmholtz free energy, which may be considered to act as a driver for emittance growth.

The topic of reversible and irreversible emittance growth is related to that of reversible and irreversible beam dynamics, which have been addressed previously both theoretically [2–4] and experimentally [5,6] for special cases. In previous work the onset of irreversible dynamics has been described in terms of the cessation of laminar particle motion as manifested by trajectory crossing in phase space in the infinite resolution limit. The possibility that rms emittance growth resulting from time-dependent space-charge forces in rf photoinjectors could be reversible under certain circumstances was proposed by Carlsten [3]. The validity of this proposal has been demonstrated experimentally both indirectly [5] and directly [6]. There are many other forms of emittance growth which may under certain circumstances be reversible. In addition to the aforementioned space-charge effects, we have a number of other forces, e.g., time-dependent rf fields, nonlinear focusing, wake fields, and coherent synchrotron radiation in bends. The purpose of this paper is to develop a general formalism. Therefore, we do not discuss the details of specific emittance growth mechanisms. A comprehensive treatment of emittance growth mechanisms can be found in Ref. [7].

The rms emittance is a practical, and widely used, figure of merit for beam optical quality. We define the normalized rms emittance of a bunch as [8,9] $\tilde{\varepsilon}_{n,x}(z) = (1/mc)\sqrt{\langle x^2\rangle\langle p_x^2\rangle - \langle x p_x\rangle^2}$, where z is the laboratory coordinate of the beam centroid, and x and p_x are conjugate spatial and momentum coordinates, respectively, the brackets represent averages over the particle distribution, m is the particle mass, and c is the velocity of light. Similar expressions can be written for the y and z emittances. It is well known that $\tilde{\varepsilon}_{n,x}$ is invariant under time-independent linear symplectic transformations [10], but not necessarily in other deterministic processes such as when nonlinear forces, or forces that are correlated with the longitudinal position in the particle bunch, are present.

In Sec. II we begin our analysis by introducing an entropy concept suitable for use with bunched beams where the resolution of a measuring apparatus is considered. In Sec. III we develop a variant of the relationship between the entropy and the rms emittance of the bunch, in which we consider the bunch to comprise a number of sub-bunches or slices in time or in longitudinal spatial coordinates. In Sec. IV, we introduce the concept of an emittance correlation coefficient C as a measure of the degree in which differential rotation of the phase space of individual slices resulting from time-dependent forces contributes to the total emittance of the bunch. We show that, even when increasing external emittance forces are included, a generalized free-energy principle is valid. We use these concepts to develop a formalism for emittance growth both with and without entropy growth in Secs. V and VI.

II. ENTROPY CONCEPT

The general connection between emittance and entropy was made over two decades ago by Lawson, Lapostolle, and Gluckstern [11]. More recently, the connection between emittance growth and entropy growth has been noted by a number of other authors [4,8,12–16], and has been developed in connection with beams where non-Hamiltonian stochastic processes are important and where the external forces are linear and time independent [4].

In equilibrium thermodynamics, entropy is considered to be either a macroscopic quantity or a microscopic quantity of a statistical ensemble. In the macroscopic thermodynamic concept, entropy cannot be defined for a system that is not in thermal equilibrium. In general, beams in linear accelerators

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are not in thermal equilibrium [17], and the nonequilibrium region may extend from the cathode to the beam dump. In this paper we will consider only the statistical interpretation of entropy as a measure of the information available about a distribution.

In the most general sense the statistical entropy of a system can be written [18]

$$S=-k_B\sum_i f_i\ln(f_i), \tag{1}$$

where k_B is Boltzmann's constant, i denotes a microcanonical state of the system, f_i is the statistical probability of that state, and $\Sigma_i f_i=1$.

In the case of a bunched beam with a very large number of particles (N) per bunch we can write the sum as an integral by noting that a state corresponds to a six-dimensional volume element $A_6=\delta x\,\delta p_x\,\delta y\,\delta p_y\,\delta\zeta\,\delta p_\zeta$, and the probability is equivalent to the product of A_6 times the distribution function $\rho_6=\rho(x,p_x,y,p_y,\zeta,p_\zeta)$, so that

$$S_6=-k_BN\int \rho_6\ln[A_6\rho_6]dx\ dp_x dy\ dp_y d\zeta\ dp_\zeta \tag{2}$$

where we choose the normalization $\int\rho_6(x)d^6x=1$. The time derivative of the entropy is given by $\dot S=-k_BN\int\dot\rho_6\ln[A_6\rho_6]dx\,dp_x dy\,dp_y d\zeta\,dp_\zeta$. Therefore the entropy of a distribution is invariant in a Hamiltonian system in which Liouville's theorem is valid, *if the variation of* ρ_6 *across the cell is negligible.*

How should we choose the size of the volume element A_6? The quantum-mechanical lower bound is $\hbar^3$, which is an unrealistically small value for a practical system. Such a volume element would have a two-dimensional normalized emittance of 3.8×10^{-7} mm mrad. Therefore, we should choose a larger value of A_6 based on the limits of our ability to make observations of the beam, i.e., on the resolution of our instrumentation, on the precision with which we can apply external forces, or on the limits of physical phenomena of interest. It is possible to consider that an apparatus designed to measure emittance by mapping the phase-space distribution could also be used to measure entropy. Such devices include so-called pepper pots, and slit-and-collector devices [19]. A suitable choice for A_6 might relate to the resolution of such an apparatus. Therefore, we assume that we have no knowledge of changes in the phase-space density on scales smaller than A_6. All techniques for measuring phase-space distributions involve segmentation of data into bins analogous to A_6, or its lower-dimensional analog. If we identify $A_6=\Delta x\Delta p_x\Delta y\Delta p_y\Delta\zeta\Delta p_\zeta$, [or as we will discuss later its two-dimensional (2D) analog $A_2=\Delta x\Delta p_x$], with the resolution of the measuring apparatus, then ρ_6 represents the averaged value over each cell. For the subsequent analysis to be meaningful, one must chose the size of the volume element to be much less than the volume occupied by the beam.

Whether or not a measured entropy change occurs in a specific case will depend on our choice of A_6, i.e., on how coarsely we divide phase space. If our apparatus had infinitesimal resolution and infinite dynamic range, then there would be no possible entropy change for a Hamiltonian system. Because of the coarse graining of the distribution function, it is possible for a process that is strictly Hamiltonian in nature to result in an entropy change if a larger, rather than a smaller, value of A_6 is chosen [12]. This concept will be important in determining whether or not emittance growth is reversible in a particular situation.

We may also consider the cell size concept in relation to the trajectory-crossing condition for the onset of irreversible dynamics of previous work [2,3]. Trajectories with an initial separation ΔR in phase space which later have zero separation are said to have crossed. If we consider two trajectories with initial separation $\Delta R>\sqrt[6]{A_6}$, that later have a separation $\Delta R\leqslant\sqrt[6]{A_6}$, then the entropy will appear to have grown. Setting $A_6=0$ in the entropy formalism results in an irreversibility criterion equivalent to the nonlaminarity condition.

III. RELATIONSHIP BETWEEN ENTROPY AND RMS EMITTANCE

We now proceed to develop a useful relationship between entropy and rms emittance. In the analysis that follows, we will develop a formalism that allows for rms emittance growth both with and without entropy growth.

When considering bunched beams with longitudinal as well as transverse structures, it is useful to divide the bunch into a number of sub-bunches or slices of longitudinal length $\delta\zeta$ in the laboratory frame, where $\delta\zeta\ll\sigma_b$ the bunch length, and ζ is the longitudinal coordinate of the sub-bunch relative to the centroid of the bunch. Consider each sub-bunch to contain a large number of particles, N_ζ. The ensemble entropy of the bunch is the sum of the entropies of the sub-bunches, i.e., $S(z)=\Sigma_\zeta S_\zeta(z)$, where S_ζ is the entropy of the sub-bunch, and z is the coordinate of the bunch centroid relative to the laboratory. Furthermore, we assume that the longitudinal momentum spread δp_ζ within each slice is small relative to the average momentum of the slice. If the longitudinal density distribution is $\rho(\zeta,p_\zeta)$, then $N_\zeta=\delta\zeta\,\delta p_\zeta\rho(\zeta,p_\zeta)$. The slice entropy and the rms emittance can be readily evaluated for specific distribution functions. If we consider a specific class of distribution functions where the four-dimensional distribution function of each slice can be written in separable form such that $\rho(x,p_x,y,p_y)=\rho_x(x,p_x)\rho_y(y,p_y)$, then, from Eq. (2), $S_\zeta(z)=S_{\zeta,x}(z)+S_{\zeta,y}(z)$, where

$$S_{\zeta,x}(z)=-k_B\delta\zeta\,\delta p_\zeta\rho(\zeta,p_\zeta)\int\rho_y(y,p_y)dy\ dp_y \times\int\rho_x(x,p_x)\ln\left(\frac{\rho_x(x,p_x)}{A_x}\right)dx\ dp_x,$$

$$S_{\zeta,y}(z)=-k_B\delta\zeta\,\delta p_\zeta\rho(\zeta,p_\zeta)\int\rho_x(x,p_x)dx\,dp_x \times\int\rho_y(y,p_y)\ln\left(\frac{\rho_y(y,p_y)}{A_y}\right)dy\ dp_y,$$

and

$$\delta x\,\delta p_x=A_x,\quad \delta y\,\delta p_y=A_y.$$

For clarity, in this paper we will assume cylindrical symmetry such that, $S_\zeta(z)=2S_{\zeta,x}(z)$, $A_x=A_y=A_2$, and that $\tilde\varepsilon_{x,n}$

TABLE I. Distribution parameter (D) for some common distribution functions.

Distribution	$(r=x^2+x'^2)$		Distribution parameter (D)
Gaussian	$e^{-r^2/2\sigma^2}$		$2e\pi \approx 5.44\pi$
Parabolic	$1-\frac{r^2}{a^2}$	$r \leqslant a$	
	0	$r>a$	5π
Uniform	1	$r \leqslant a$	
	0	$r>a$	4π
Hollow	$r^2 e^{-r^2/2\sigma^2}$		3.105π

$=\tilde{\varepsilon}_{y,n}=\tilde{\varepsilon}_{\zeta,n}(\zeta,z)$. The number of occupied cells in 2D phase space is given by $\exp(S_\zeta/2k_B N_\zeta)$ [14]. Therefore the total phase-space area (Σ) occupied by the slice is $\Sigma = A_2 \exp(S_\zeta/2k_B N_\zeta)$. For a given distribution function, we can write $\Sigma/\tilde{\varepsilon}_{\zeta,n}=D$, where D is a unitless distribution-dependent parameter which may vary from slice to slice. Therefore, we can write the relationship between the slice entropy and the slice emittance as

$$S_\zeta = 2k_B N_\zeta \ln\left(\frac{D\tilde{\varepsilon}_{\zeta,n}}{A_2}\right). \tag{3}$$

We can evaluate D for specific distributions, as shown in Table I. We expect D to be largest for a Gaussian distribution because the maximum entropy state corresponds to that of an equilibrium Maxwell-Boltzmann distribution, which will have a Gaussian distribution in phase space when space-charge forces are not important [20,21].

Because the entropy of the bunch is simply a sum of the subentropies, the bunch entropy does not depend on the orientation of the individual slices in phase space. Therefore, it is useful to introduce a quantity called the entropy of the average slice $S_{\langle\zeta\rangle}$, which is defined with respect to the average slice emittance of the bunch as

$$S_{\langle S\rangle}(z) = 2k_B N \ln\left(\frac{D\langle\tilde{\varepsilon}_{\zeta,n}\rangle}{A_2}\right), \tag{4}$$

where $\langle\tilde{\varepsilon}_{\zeta,n}\rangle = \int \tilde{\varepsilon}_{\zeta,n}\rho(\zeta)d\zeta$ is the slice emittance averaged over the bunch.

$S_{\langle\zeta\rangle}$ has similar properties to the total entropy of the bunch. In particular, $\Delta S_{\langle\zeta\rangle}>0$ when $\Delta S>0$, and $\Delta S_{\langle\zeta\rangle}=0$ when $\Delta S=0$.

IV. DIFFERENTIAL PHASE-SPACE ROTATION AND THE CORRELATED EMITTANCE COEFFICIENT

In our formalism, we consider transverse emittance growth to have two components: *time dependent* and *time independent*. The time-dependent component involves the differential rotation of the slices in phase space. In a time-dependent transformation the orientation of the individual slices with respect to each other is invariant. For time-dependent transformations the relative orientations of the slices may change, and result in a growth in the rms emittance of the bunch. Various effects such as nonuniform longitudinal space-charge distributions, transverse wake fields, and phase-dependent rf forces will result in the twisting of the phase space being correlated with position of the individual slices in the bunch. The second, time-independent component of emittance growth results from growth in the emittance of individual slices. Such time-independent growth results from nonlinear spatial forces.

We assume, for convenience, that on scales shorter than the slice length $\delta\zeta$ that the transverse phase space is uncorrelated with ζ, i.e., we have no knowledge of the details of the time dependence of the distribution on scales shorter than $\delta\zeta$. On scales longer than $\delta\zeta$, we assume that there may be correlations between the phase space and the location of the slice in the bunch.

The phase-space orientation of the ith slice may be characterized by Twiss parameters: $\hat{\alpha}_i=\hat{\alpha}(\zeta,z)$, $\hat{\beta}_i=\hat{\beta}(\zeta,z)$, and $\hat{\gamma}_i=\hat{\gamma}(\zeta,z)$, with $\hat{\alpha}_i^2+1=\hat{\beta}_i\hat{\gamma}_i$. We envision processes where the phase-space distribution of the beam evolves such that the Twiss parameters change in a fashion determined by ζ-dependent forces. If the forces that determine the phase-space evolution are correlated in ζ (i.e., nonstochastic), then the phase-space evolution will be correlated in ζ also. The emittance change of each sub-bunch is determined by nonlinear and stochastic transverse forces.

We can write the normalized rms emittance of the bunch for the x coordinate in terms of the slice emittances and slice Twiss parameters as

$$\begin{aligned}
&\tilde{\varepsilon}_n^{\,2}(z) \\
&= \int \tilde{\varepsilon}_{\zeta,n}(\zeta,z)\hat{\beta}(\zeta,z)\rho(\zeta)d\zeta \int \tilde{\varepsilon}_{\zeta,n}(\zeta,z)\hat{\gamma}(\zeta,z)\rho(\zeta)d\zeta \\
&\quad -\left(\int \tilde{\varepsilon}_{\zeta,n}(\zeta,z)\hat{\alpha}(\zeta,z)\rho(\zeta)d\zeta\right)^2 \\
&= \langle\tilde{\varepsilon}_{\zeta,n}(\zeta,z)\hat{\beta}(\zeta,z)\rangle\langle\tilde{\varepsilon}_{\zeta,n}(\zeta,z)\hat{\gamma}(\zeta,z)\rangle \\
&\quad -\langle\tilde{\varepsilon}_{\zeta,n}(\zeta,z)\hat{\alpha}(\zeta,z)\rangle^2.
\end{aligned} \tag{5}$$

To aid in distinguishing between time-dependent and time-independent emittance growth, we define an rms emittance correlation coefficient C such that $C^2(z)=\tilde{\varepsilon}_n^{\,2}/\langle\tilde{\varepsilon}_{\zeta,n}\rangle^2$. Since $\tilde{\varepsilon}_n(z)\geqslant\tilde{\varepsilon}_{\zeta,n}$ always, $C(z)\geqslant 1$ always. When all the equivalent phase-space ellipses of the slices are aligned, $C=1$, its minimum value. We consider the phase-space slice to be aligned when the Twiss parameters are independent of ζ, such that $\hat{\alpha}$, and $\hat{\beta}$, and $\hat{\gamma}$ can be taken outside the average operators in Eq. (5) resulting in $\tilde{\varepsilon}_n(z)=\tilde{\varepsilon}_{\zeta,n}$. When the ellipses are not aligned, $C>1$, because of the differential expansion of the slices and rotation of each phase-space slice with respect to its neighbor. Differentiating $C(z)$ with respect to z gives

$$\frac{d\tilde{\varepsilon}_n^{\,2}}{dz} = \langle\tilde{\varepsilon}_{\zeta,n}\rangle^2 \frac{dC^2}{dz} + 2\tilde{\varepsilon}_n^2(z) \frac{d(\ln\langle\tilde{\varepsilon}_{\zeta,n}\rangle)}{dz}. \tag{6}$$

Changes in the total rms emittance of the bunch occur either from changes in the phase-space correlations or from changes in the slice emittances. It is also possible for $\tilde{\varepsilon}_n$ to remain constant while C decreases, and for the slice emit-

tance to increase as the beam is transported through the accelerator. It is now appropriate to examine this in the context of entropy changes. Substituting Eq. (4) into Eq. (6), we obtain

$$\frac{d\tilde{\varepsilon}_n^2}{dz}=\langle\tilde{\varepsilon}_{\zeta,n}\rangle^2\frac{dC^2}{dz}-\frac{2\tilde{\varepsilon}_n^2}{D}\frac{dD}{dz}+\frac{\tilde{\varepsilon}_n^2}{k_BN}\frac{d(S_{\langle\zeta\rangle})}{dz}. \quad (7)$$

Let us consider the physical interpretation of Eq. (7). We will consider two cases: one where the entropy change is zero, the other where the entropy change is nonzero.

V. EMITTANCE CHANGE WITHOUT ENTROPY CHANGE: NONLINEAR ENERGY FUNCTION

If $d(S_{\langle\zeta\rangle})/dz=0$, i.e., zero entropy change, we have from Eq. (7), and the definition of C that $(\tilde{\varepsilon}_n^2/D)/(dD/dz)=-(C^2/2)/(d\langle\tilde{\varepsilon}_{\zeta,n}\rangle^2/dz)$. We assume, in this case, that the forces that drive the phase-space evolution are smoothly varying in space and time, i.e., no significant stochastic forces are present, and we are in the infinite resolution limit.

In the following analysis we derive a differential equation for the rms emittance change under general time-dependent and nonlinear forces. Note that we use emittance *change* rather than just emittance *growth*. The emittance may increase or decrease depending on the details of the beam dynamics.

We now consider a general zero-entropy-change case, corresponding to infinite resolution ($A_2=0$), in which where each slice is moving under transverse space-charge forces $qE_{sc}(x,\zeta,z)$, externally applied time-dependent forces $qE_x(x,\zeta,z)$ (e.g., from rf fields, wake fields, etc.), external time-independent magnetic or electric focusing forces represented by $qK_x(x,z)$, and longitudinal time-dependent electric forces $qE_z(x,\zeta,z)$, such that the motion of individual particles in each slice is given by

$$x''+\frac{qE_z(\zeta,z)x'}{mc^2\beta^2\gamma}+\frac{qK_x(x,z)}{mc^2\beta^2\gamma}-\frac{qE_{sc}(x,\zeta,z)}{mc^2\beta^2\gamma^3}+\frac{qE_x(x,\zeta,z)}{mc^2\beta^2\gamma}=0, \quad (8)$$

where the prime indicates the derivative with respect to z, q is the particle charge, and $mc\beta\gamma$ is the longitudinal momentum. In Eq. (8) and subsequent analysis, we assume that $x'\ll 1$. We can determine the change of emittance with longitudinal distance by differentiating Eq. (7) to obtain

$$\frac{d\tilde{\varepsilon}_n^2(z)}{dz}=\langle\tilde{\varepsilon}_{\zeta,n}\hat{\beta}'\rangle\langle\tilde{\varepsilon}_{\zeta,n}\gamma\rangle+\langle\tilde{\varepsilon}_{\zeta,n}\hat{\beta}\rangle\langle\tilde{\varepsilon}_{\zeta,n}\hat{\gamma}'\rangle-2\langle\tilde{\varepsilon}_{\zeta,n}\hat{\alpha}\rangle\times\langle\tilde{\varepsilon}_{\zeta,n}\hat{\alpha}\hat{\alpha}'\rangle+\langle(\tilde{\varepsilon}_{\zeta,n})'\hat{\beta}\rangle\langle\tilde{\varepsilon}_{\zeta,n}\hat{\gamma}'\rangle+\langle\tilde{\varepsilon}_{\zeta,n}\hat{\beta}\rangle\times\langle(\tilde{\varepsilon}_{\zeta,n})'\hat{\gamma}\rangle-2\langle\tilde{\varepsilon}_{\zeta,n}\hat{\alpha}\rangle\langle(\tilde{\varepsilon}_{\zeta,n})'\hat{\alpha}\rangle, \quad (9)$$

where $\tilde{\varepsilon}_{s,n}$ and the Twiss parameters are functions of both ζ and z. By noting that

$$\hat{\alpha}'=\frac{d\left\{\frac{\gamma\langle xx'\rangle_\zeta}{\tilde{\varepsilon}_{\zeta,n}}\right\}}{dz},\quad \hat{\beta}'=\frac{d\left\{\frac{\gamma\langle x^2\rangle_\zeta}{\tilde{\varepsilon}_{\zeta,n}}\right\}}{dz},$$

and

$$\hat{\gamma}'=\frac{d\left\{\frac{\gamma\langle x'x'\rangle_\zeta}{\tilde{\varepsilon}_{\zeta,n}}\right\}}{dz},$$

and using Eq. (8), we may evaluate the Twiss parameters for each slice as

$$\hat{\alpha}'=\hat{\gamma}-\frac{q}{mc^2\beta\tilde{\varepsilon}_{\zeta,n}}\langle xK_x\rangle_\zeta+\frac{q}{mc^2\beta\gamma^2\tilde{\varepsilon}_{\zeta,n}}\langle xE_{sc}\rangle_\zeta-\tilde{\varepsilon}_{\zeta,n}\langle xE_x\rangle_\zeta-\frac{(\tilde{\varepsilon}_{\zeta,n})'}{\tilde{\varepsilon}_{\zeta,n}}\hat{\alpha},$$

$$\hat{\beta}'=2\alpha+\left(\frac{qE_z}{mc^2\beta^2\gamma}-\frac{(\tilde{\varepsilon}_{\zeta,n})'}{\tilde{\varepsilon}_{\zeta,n}}\right)\hat{\beta},$$

$$\hat{\gamma}'=-\frac{2}{mc^2\beta\tilde{\varepsilon}_{\zeta,n}}\langle x'K_x\rangle_\zeta+\frac{2q}{mc^2\beta\gamma^2\tilde{\varepsilon}_{\zeta,n}}\langle x'E_{sc}\rangle_\zeta-\frac{2q}{mc^2\beta\tilde{\varepsilon}_{\zeta,n}}\langle x'E_x\rangle_\zeta-\left(\frac{qE_z}{mc^2\beta^2\gamma}+\frac{(\tilde{\varepsilon}_{\zeta,n})'}{\tilde{\varepsilon}_{\zeta,n}}\right)\hat{\gamma},$$

where $\langle\ \rangle_\zeta$ indicates an average over a slice. After some straightforward algebra, we can rewrite Eq. (9) as

$$\frac{d\tilde{\varepsilon}_n^2(z)}{dz}=\frac{2q}{mc^2\gamma}\left[\begin{array}{l}\langle x'E_{sc}\rangle\langle x^2\rangle-\langle xx'\rangle\langle xE_{sc}\rangle\\-\gamma^2(\langle x'E_x\rangle\langle x^2\rangle-\langle xx'\rangle\langle xE_x\rangle+\langle x'K_x\rangle\langle x^2\rangle-\langle xx'\rangle\langle xK_x\rangle)\end{array}\right]$$

$$=\langle\tilde{\varepsilon}_{\zeta,n}\rangle^2\frac{dC^2}{dz}+C^2\frac{d\langle\tilde{\varepsilon}_{\zeta,n}\rangle^2}{dz}, \quad (10)$$

where $\langle\ \rangle$ represents the average over the entire bunch. We identify changes in C as resulting from time-dependent forces.

We note that the focusing forces may be decomposed into a sum of two terms: the first is the linear, phase-independent term; the second contains the nonlinear and the phase-dependent terms, i.e., $E_{sc}=a_1x+E_{sc}^*$, $E_x=a_2x+E_x^*$, and $K_x=a_3x+K_x^*$, where * indicates a term which is nonlinear in x and/or phase dependant, and a_1, a_2, and a_3 are constants. Therefore, the terms on the right-hand side of Eq. (10) can be rewritten as

$$\langle x'E_{sc}\rangle\langle x^2\rangle-\langle xx'\rangle\langle xE_{sc}\rangle=\langle x'E_{sc}^*\rangle\langle x^2\rangle-\langle xx'\rangle\langle xE_{sc}^*\rangle$$
$$=-\frac{\langle x^2\rangle}{qN}\frac{d[U(z)-U_0(z)]}{dz},$$

$$\langle x'E_x\rangle\langle x^2\rangle-\langle xx'\rangle\langle xE_x\rangle=\langle x'E_x^*\rangle\langle x^2\rangle-\langle xx'\rangle\langle xE_x^*\rangle$$
$$=\frac{\langle x^2\rangle}{qN}\frac{d[V(z)-V_0(z)]}{dz},$$

$$\langle x'K_x\rangle\langle x^2\rangle-\langle xx'\rangle\langle xK_x\rangle=\langle x'K_x^*\rangle\langle x^2\rangle-\langle xx'\rangle\langle xK_x^*\rangle$$
$$=\frac{\langle x^2\rangle}{qN}\frac{d[W(z)-W_0(z)]}{dz},$$

so that

$$d\frac{\tilde{\varepsilon}_n^2(z)}{dz}=\frac{-2\langle x^2\rangle\gamma}{mc^2N}\left[\frac{1}{\gamma^2}\frac{d[U(z)-U_0(z)]}{dz}+\frac{d[V(z)-V_0(z)]}{dz}+\frac{d[W(z)-W_0(z)]}{dz}\right]. \tag{11}$$

The expressions in Eq. (11) correspond to the emittance growth resulting from self-field and external-field energy changes, where U is the space-charge field energy of the beam, U_0 is the space-charge field energy of the equivalent uniform beam, V is the transverse kinetic energy of the bunch induced by the external time-dependent forces, V_0 is the kinetic energy that would have been induced if the transverse rf forces were linear in x and phase independent, W is the transverse kinetic energy of the bunch induced by the external time-independent focusing forces, and W_0 is the kinetic energy that would have been induced if the external focusing forces were linear in x. The portion of the expression in Eq. (11) involving the space-charge field energy has been derived previously [2,22–24] for the limited case of drifting beams with linear time-independent external focusing fields. We see that emittance growth in the zero-entropy-change case can be characterized by a general energy principle in the form of Eq. (11). In the case where no emittance growth occurs, expansion and contraction of the bunch result in simple exchanges between potential and kinetic energy.

The terms on the right hand side of Eq. (11) are similar in form; therefore we may simplify our notation by defining a general nonlinear energy function $\hat{U}$ such that

$$\hat{U}'(z)=\frac{1}{\gamma^2}\frac{d[U(\zeta)-U_0(z)]}{dz}+\frac{d[V(z)-V_0(z)]}{dz}+\frac{d[W(z)-W_0(z)]}{dz}.$$

Then we can write a simpler form of Eq. (10) as $\tilde{\varepsilon}_n'(z)=[-\langle x^2\rangle\gamma/mc^2N\tilde{\varepsilon}_n(z)]\hat{U}'(z)$. If a beam distribution is matched in to a transport system such that the right hand side of Eq. (11) is zero, then no emittance growth occurs. Therefore, matching of a beam into a transport system simply requires that $\hat{U}'=0$.

The emittance changes described by Eq. (11) are deterministic, not stochastic, and can be positive or negative depending on the details of the beam dynamics. The time-dependent forces give rise to changes in the correlated emittance coefficient C, and the forces that are nonlinear in x give rise to the slice emittance growth. Application of appropriate forces, such as the inverse of the forces that caused emittance growth, will result in removal of the emittance growth if no entropy growth has occurred.

A practical demonstration of emittance growth reversal can be found in rf electron photoinjectors where solenoidal emittance compensation is used [3]. In this case space-charge forces and rf forces combine to introduce correlations into the bunch, and result in large emittance growth over the first few centimeters of beam acceleration. Because of rapid cooling of the bunch in the longitudinal direction during acceleration, initially there is a negligible diffusion of the particles from slice to slice. The emittance growth may be removed by appropriate focusing of the bunch [3,25], resulting in electron beams of unprecedented brightness [5,26,27], i.e., $C(z)$ grows and can be brought back close to unity when appropriate focusing forces are applied.

VI. EMITTANCE GROWTH WITH ENTROPY CHANGE: GENERALIZED FREE ENERGY FUNCTION

We now consider cases where the entropy of the bunch changes. We note that both C and the slice emittance can increase or decrease. The entropy of the distribution, however, will tend to increase. Such irreversible events can be driven by such effects as Coulomb collisions and thermal diffusion [28,29], space-charge wave breaking [2,30,31], rf noise, or phase-space filamentation on the scale of the cell size A_2 [12].

We can include the entropy change in our expression for emittance growth by combining Eqs. (7) and (10) and rearranging terms. This results in a generalized expression for

the emittance growth that involves both the reversible and irreversible components:

$$\tilde{\varepsilon}_n' = \frac{-\gamma\langle x^2\rangle}{mc^2 N\tilde{\varepsilon}_n}\left(\frac{1}{\gamma^2}\frac{d[U-U_0]}{dz}+\frac{d[V-V_0]}{dz}+\frac{d[W-W_0]}{dz}-T_{\mathrm{ef}}\frac{dS_{\langle\zeta\rangle}}{dz}\right)$$

$$=\frac{-\gamma\langle x^2\rangle}{mc^2 N\tilde{\varepsilon}_n}(\hat{U}'-T_{\mathrm{ef}}S'_{\langle\zeta\rangle})$$

$$=\frac{-\gamma\langle x^2\rangle}{mc^2 N\tilde{\varepsilon}_n}\hat{F}', \tag{12}$$

where $T_{\mathrm{ef}}=(mc^2\tilde{\varepsilon}_n^{\,2}/k_B\gamma\langle x^2\rangle)$ is the effective transverse temperature of the beam in the laboratory frame [7], and $\hat{F}'=\hat{U}'-T_{\mathrm{ef}}S'_{\langle\zeta\rangle}$. Note that the above definition of T_{ef} does not imply that the phase space distribution is in thermal equilibrium. We have defined a generalized free-energy function $\hat{F}$, which is analogous to the familiar Helmholtz free energy.

Analysis of the generalized free-energy function provides insight into phenomena that have been noted in previous work. The system (in this case the phase space of the bunch) will evolve so that $\hat{F}'$ moves toward a zero value, i.e., the emittance growth or reduction stops. Changes in T, for constant rms emittance, correspond to expansions and contractions of the bunch. In the case of a cold beam, where the space charge dominates over the emittance, and in the absence of external nonlinear forces, the $\hat{U}$ term can be much greater than the TS term, and the distribution will tend toward uniformity [2,22–24], its lowest energy state. (We note that for $T=0$ the appropriate Maxwell-Boltzmann distribution is a uniform distribution [17]). In the case of a hot, i.e., emittance-dominated beam, the TS term dominates, leading to a Gaussian transverse distribution such as in high energy storage rings [20,21] where space-charge forces are negligible. In the case where nonlinear external forces dominate, the distribution function should relax to a form appropriate to that force, such that $d[W(z)-W_0(z)]/dz=0$, and such that the emittance growth stops, i.e., the beam distribution has been transformed in to a matched distribution for the transport system. Furthermore, if the beam distribution is in thermal equilibrium, and matched into the transport system (i.e., $\hat{U}'=0$), then no entropy growth can occur (i.e., $S'=0$) and hence no emittance growth occurs.

In practice, we can consider emittance growth to have two components: the reversible part where $\Delta S=0$, and the irreversible part where $\Delta S>0$, such that $\Delta\tilde{\varepsilon}(z)^2=\Delta\tilde{\varepsilon}(z)_I^2 \pm\Delta\tilde{\varepsilon}(z)_R^2$, where the subscripts R and I stand for reversible and irreversible, respectively. A determination as to what portion of the emittance growth is reversible in a particular process can be made by evaluating the entropy change using Eq. (2). Whether or not entropy growth has occurred will depend, among other things, on the choice of A and $\delta\zeta$.

There is no guarantee that reversible emittance growth can in fact be reversed. This is because, in the finite resolution case, effective entropy growth can occur in the absence of stochastic phenomena within the beam. The degree of achievable reversibility depends on our ability to apply corrective forces to the bunch. For example, it is possible to measure the phase-space distribution with a fine resolution, and to conclude that no entropy growth has occurred. If, however, we are unable to apply corrective forces on this fine scale, we should conclude that effective entropy growth has occurred and should recalculate the entropy using a coarser scale. Therefore, our choice of A_2 and $\delta\zeta$ should be made with reference to the spatial and temporal resolution of our corrective apparatus.

A question which is related to reversibility is as follows: When presented with a beam of given phase-space distribution and emittance, what are the criteria for determining whether or not the emittance may be reduced by the application of deterministic forces? Based on our previous discussion there appear to be two conditions under either of which the emittance may be reduced: (1) if the correlated emittance coefficient $C>1$; and (2) if the distribution parameter $D<2e\pi$, where $D=[A_2\exp(S_{\langle\zeta\rangle}/2k_BN)]/\langle\tilde{\varepsilon}_{s,n}\rangle$.

Under condition (1), the emittance reduction may be accomplished by applying time-dependent forces. Under condition (2), for emittance reduction to occur, D must increase while the emittance decreases in order to keep $S_{\langle s\rangle}$ constant. For example, in the latter case, take a space-charge-dominated beam that has an initially uniform distribution and focus it nonlinearly, so that it becomes more Gaussian in form.

Another consequence of the analysis presented here is that entropy growth does not necessarily imply emittance growth. We see from Eq. (11) that changes in entropy can, in principle, be offset or overcome by the application of appropriate nonlinear focusing forces.

VII. CONCLUSION

A detailed analysis of the time scales for the onset of irreversible dynamics and entropy growth in particular cases is beyond the scope of this paper. An approach using the Fokker-Planck equation [4,22,28–32] may be adapted to determine the stochastic entropy growth rate in certain cases. A useful approach for future work will be to use particle simulations to evaluate the entropy changes in various situations of interest directly, so as to determine whether or not phase-space evolution can be described by an equation such as Eq. (11), and to access the validity of other predictions of the theory presented herein. Recent progress in the experimental mapping of phase space on a submicrobunch scale offers the possibility of the testing of some of the concepts outlined in this paper.

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